# ECED 3300
# Tutorial 3

## Problem 1

*Verify Gauss's theorem for the field* $\mathbf{F} = \mathbf{a}_\rho \rho + \mathbf{a}_z z$ *and a quarter cylinder,* $0 \le \phi \le \pi/2$, $0 \le z \le h$ *and* $0 \le \rho \le R$.

### Solution

1) Bottom surface: $z = 0$, $\mathbf{a}_n = -\mathbf{a}_z$, $\mathbf{F}(z=0)\cdot\mathbf{a}_n = 0$; Top surface: $\mathbf{a}_n = \mathbf{a}_z$, $\mathbf{F}(z=h)\cdot\mathbf{a}_n = h$, $dS = \rho d\rho d\phi$.

2) Walls: $\mathbf{a}_n = \mathbf{a}_\rho$, $\mathbf{F}(\rho = R)\cdot\mathbf{a}_n = R$, $dS = Rd\phi dz$.

3) End faces, $\phi = 0$ and $\phi = \pi/2$, $\mathbf{a}_n = \mathbf{a}_\phi$ such that $\mathbf{F}\cdot\mathbf{a}_n = 0$. Thus,

$$\int_{top} d\mathbf{S}\cdot\mathbf{F} = \int_0^R d\rho\rho \int_0^{\pi/2} d\phi h = \pi h R^2/4$$

and

$$\int_{walls} d\mathbf{S}\cdot\mathbf{F} = \int_0^h dz\rho \int_0^{\pi/2} d\phi R^2 = \pi h R^2/2$$

Thus,

$$\oint d\mathbf{S}\cdot\mathbf{F} = \int_{top} d\mathbf{S}\cdot\mathbf{F} + \int_{walls} d\mathbf{S}\cdot\mathbf{F} = 3\pi h R^2/4.$$

On the other hand,

$$\nabla\cdot\mathbf{F} = \tfrac{1}{\rho}\partial_\rho(\rho F_\rho) + \partial_z F_z = 2 + 1 = 3.$$

$$\int dv \nabla\cdot\mathbf{F} = 3\int_0^h dz \int_0^R d\rho\rho \int_0^{\pi/2} d\phi = 3\pi h R^2/4.$$

## Problem 2

*Given* $\mathbf{F} = x^2\mathbf{a}_x + y^2\mathbf{a}_y + (z^2-1)\mathbf{a}_z$, *find* $\oint_S d\mathbf{S}\cdot\mathbf{F}$ *where the surface* $S$ *is defined by* $\rho = 2$, $0 \le z \le 2$ *and* $0 \le \phi \le 2\pi$.

### Solution

As $S$ is a **closed surface**, we can take advantage of Gauss's theorem,

$$\oint_S d\mathbf{S}\cdot\mathbf{F} = \int_v dv \nabla\cdot\mathbf{F}. \quad (1)$$

Further, the divergence is **independent** of a coordinate system. Hence using Cartesian coordinates,

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = 2(x + y + z).$$

Transforming to cylindrical coordinates, we obtain

$$\nabla \cdot \mathbf{F} = 2(x + y + z) = 2(\rho \cos\phi + \rho \sin\phi + z). \tag{2}$$

It follows from Eqs. (1) and (2) that

$$\begin{aligned}
&\oint_S d\mathbf{S} \cdot \mathbf{F} = \int_v dv \nabla \cdot \mathbf{F} = 2 \int_0^2 d\rho\rho \int_0^{2\pi} d\phi \int_0^2 dz \, (\rho \cos\phi + \rho \sin\phi + z) \\
&= 2 \left[ \int_0^2 d\rho\rho \int_0^{2\pi} d\phi \int_0^2 dz \, z + \int_0^2 d\rho\rho^2 \underbrace{\int_0^{2\pi} d\phi \cos\phi}_{=0} \int_0^2 dz + \int_0^2 d\rho\rho^2 \underbrace{\int_0^{2\pi} d\phi \sin\phi}_{=0} \int_0^2 dz \right] \\
&= 2 \frac{\rho^2}{2}\bigg|_0^2 \times 2\pi \times \frac{z^2}{2}\bigg|_0^2 = 16\pi.
\end{aligned} \tag{3}$$

# Problem 3

*One of Maxwell's equations states that any magnetic field must be solenoidal, that is $\nabla \cdot \mathbf{B} = 0$. Use this information to determine the flux of a uniform magnetic field, $\mathbf{B} = \mathbf{a}_z B$, $B = const$ through the curved surface of a right circular cone of radius $R$ and height $h$ oriented so that $\mathbf{B}$ is normal to the cone base which is located in the $xy$-plane. The cone axis coincides with the $z$-axis.*

### Solution

Since the cone is a closed surface, we can apply Gauss's theorem to it

$$\oint d\mathbf{S} \cdot \mathbf{B} = \int dv \nabla \cdot \mathbf{B} = 0.$$

Thus, the magnetic field flux trhough the **entire** cone surface must be zero. It then follows at once that $\oint d\mathbf{S} \cdot \mathbf{B} = 0 \Longrightarrow \int_{curved} d\mathbf{S} \cdot \mathbf{B} = -\int_{base} d\mathbf{S} \cdot \mathbf{B}$. Hence, figuring out the flux through the curved surface boils down to determining the flux through the base. The latter is straightforward. At $z = 0$, $\mathbf{a}_n = -\mathbf{a}_z$ and $dS = \rho d\rho d\phi$, $\mathbf{B} = \mathbf{a}_z B$. Thus,

$$\int_{curved} d\mathbf{S} \cdot \mathbf{B} = -\int_{base} d\mathbf{S} \cdot \mathbf{B} = -\int_0^{2\pi} d\phi \int_0^R d\rho\rho(-\mathbf{a}_z \cdot \mathbf{a}_z)B = \pi R^2 B.$$

# Problem 4

*Employ the divergence theorem to show that for any closed surface enclosing a volume $V$,*

$$V = \frac{1}{3}\oint_S d\mathbf{S}\cdot\mathbf{r},$$

*where $\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$ is a radius vector to an arbitrary point. Use this result to figure out the volume of*

*a) a rectangular parallelepiped with sides $a$, $b$ and $c$;*

*b) a sphere of radius $R$,*

*c) a right circular cone of height $h$ and base radius $R$.*

**Solution**

Consider the divergence theorem for $\mathbf{F} = \mathbf{r}$,

$$\oint_S d\mathbf{S}\cdot\mathbf{r} = \int dv\nabla\cdot\mathbf{r}.$$

By definition, $\nabla\cdot\mathbf{r} = \partial_x x + \partial_y y + \partial_z z = 3$. It follows that $\int dv\nabla\cdot\mathbf{r} = 3\int dv = 3V$. Finally,

$$V = \frac{1}{3}\oint_S d\mathbf{S}\cdot\mathbf{r},$$

Q.E.D.

a) Choose a Cartesian coordinate system with the origin at the parallelepiped center, $-a/2 \leq x \leq a/2$, $-b/2 \leq y \leq b/2$ and $-c/2 \leq z \leq c/2$. By symmetry, it's enough to consider just one side, $x = \pm a/2$, say. At $x = \pm a/2$, $\mathbf{a}_n = \pm\mathbf{a}_x$, $dS = dydz$. Generalizing,

$$3V = \int_{-b/2}^{b/2}\int_{-c/2}^{c/2} dydz[a/2-(-a/2)] + \int_{-a/2}^{a/2}\int_{-c/2}^{c/2} dxdz[b/2-(-b/2)] + \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} dxdy[c/2-(-c/2)] = 3abc.$$

It follows

$$V = abc.$$

b) For a sphere, $\mathbf{a}_n = \mathbf{a}_r$ and using the spherical coordinates, $\mathbf{r} = \mathbf{a}_r r$ such that on the surface, $\mathbf{r} = R\mathbf{a}_r$ and $dS = R^2\sin\theta d\theta d\phi$

$$V = \frac{1}{3}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\sin\theta R^2\times R(\mathbf{a}_r\cdot\mathbf{a}_r) = 4\pi R^3/3.$$

c) Let us place the cone so that its apex is at the origin. One then immediately observes that everywhere on the curved surface $\mathbf{r}\cdot\mathbf{a}_n = 0$. Hence, the curved surface makes no contribution to the flux. The flux of $\mathbf{r} = \mathbf{a}_\rho \rho + \mathbf{a}_z z$ through the base at $z = h$ is easy. Indeed, at $z = h$ $\mathbf{a}_n = \mathbf{a}_z$ such that $\mathbf{r} \cdot \mathbf{a}_n = h$ and $dS = \rho d\rho d\phi$. Finally,

$$V = \frac{1}{3} \int_0^{2\pi} d\phi \int_0^R d\rho \rho h = \pi R^2 h/3.$$

# Problem 5

*Show that* $\oint_S dS\mathbf{a}_n = 0$ *for any closed surface* $S$.

**Solution**

Consider an arbitrary **constant** vector $\mathbf{a}$. Take a dot product of $\mathbf{a}$ and $\oint_S dS\mathbf{a}_n$,

$$\mathbf{a} \cdot \oint_S dS\mathbf{a}_n = \oint_S dS(\mathbf{a} \cdot \mathbf{a}_n) = \oint_S d\mathbf{S} \cdot \mathbf{a} = \int dv \nabla \cdot \mathbf{a} = 0,$$

The last line follows from the fact that $\mathbf{a}$ is a constant vector. Hence we have shown that for an **arbitrary** constant vector $\mathbf{a}$,

$$0 = \oint_S dS(\mathbf{a} \cdot \mathbf{a}_n) = \mathbf{a} \cdot \oint_S dS\mathbf{a}_n.$$

It then follows that it can happen iff

$$\oint_S dS\mathbf{a}_n = 0.$$

Q.E.D.